13. Monotone Comparative Statics in Games

Duarte Gonçalves

University College London

MRes Microconomics

Overview

Fixed point theorems and comparative statics results are the bread and butter of economic theory.

But not only theory!

Some other examples of questions addressed using these tools:

- (Macro) Comparative statics on equilibrium prices and quantities when there is a demand shock induced by a change in consumers' preferences (e.g., Acemoglu & Jensen 2015 JPE).
- (Econometrics) Nonparametric partial identification of treatment response with social interactions (e.g.Lazzati (2015 QE), with an application to studying the effect of police per capita on crime rates).
- (Health) Empirical antitrust implications of centralised matching systems on wages of medical residents (Agarwal 2015 AER).
- (Education) The empirical consequences of affirmative action in university admission (Dur, Pathak, & Sonmez 2020 JET; Aygun & Bo 2021 AEJMicro).

Overview

Agenda for today:

1. Two new fixed-point theorems based on monotonicity conditions.
2. Strong and weak monotone comparative statics of fixed points.

Overview

Overview

Strong Set Order

$(X, \geq)$ lattice.

Recall: **strong set order** $\geq_{SS}$, a binary relation on $\mathbf{2}^X$:

Definition

S' **strong set dominates** S ($S' \geq_{SS} S$) if $\forall x' \in S'$, $x \in S$, $x \vee x' \in S'$ and $x \wedge x' \in S$.

Strong set order can be too demanding (and therefore inapplicable) to many situations.

E.g., for comparative statics on equilibria (sets of fixed points) when some fundamental changes.

Strong Set Order

Players I and each player i can choose strategies s_i in S_i.

Choices given by $B_i : S_{-i} \times \boldsymbol{\Theta}_i \rightrightarrows S_i$. Choices depend on opponents' choices and some parameter $\boldsymbol{\theta}_i$.

Fixed points (equilibria) $\mathcal{F}(B, \boldsymbol{\theta})$, $s_i \in B_i(s_{-i}, \boldsymbol{\theta}_i)$ for every $i \in I$, and how they depend on $\boldsymbol{\theta}$. Fixed point: choices of different players are consistent.

Hardly ever going to have strong-set ordered equilibria:

If $s \in \mathcal{F}(B, \boldsymbol{\theta})$ and $s' \in \mathcal{F}(B, \boldsymbol{\theta}')$, quite demanding to ask that $s \vee s'$ and $s \wedge s'$ are also equilibria!

Weak Set Order

A less stringent way of ordering sets: weak set order (see Che, Kim, & Kojima 2021 WP).

Definition

(i) S' **upper weak set dominates** S ($S' \geq_{uws} S$) iff $\forall x \in S, \exists x' \in S'$ s.t. $x' \geq x$;

(ii) S' **lower weak set dominates** S ($S' \geq_{lws} S$) iff $\forall x' \in S', \exists x \in S$ s.t. $x' \geq x$;

(iii) S' **weak set dominates** S ($S' \geq_{ws} S$) iff S' both upper and weak set dominates S; i.e., $\geq_{ws} = \geq_{uws} \cap \geq_{lws}$.

Weak Set Order

How do strong and weak set orders compare?

Lemma

(i) $\geq_{ss}$ is transitive and antisymmetric on non-empty sets. It is not necessarily either reflexive or irreflexive.

(ii) $\geq_{ws}$ is transitive and reflexive, but not necessarily antisymmetric.

(iii) $\forall$ nonempty subsets $S, T \subseteq X$, $S \geq_{ss} T \implies S \geq_{ws} T$. (i.e., $\geq_{ss} \subseteq \geq_{ws}$.)

(iv) $\geq_{ss}$ is closed under intersection, i.e., $\forall$ non-empty $S, S', T, T' \subseteq X$ s.t. $S' \geq_{ss} S$ and $T' \geq_{ss} T$, $S' \cap T' \geq_{ss} S \cap T$. It is not necessarily closed under union.

(v) $\geq_{ws}$ is closed under union, i.e., $\forall$ non-empty $S, S', T, T' \subseteq X$ s.t. $S' \geq_{ws} S$ and $T' \geq_{ws} T$, $S' \cup T' \geq_{ws} S \cup T$. It is not necessarily closed under intersection.

Exercise

Overview

Monotone Mappings

Before we do comparative statics: Tarski's Fixed-Point Theorem.

Arguably one of the most useful fixed point theorems.

Definition

Function $f : X \to X$ is **monotone** iff it is order-preserving, i.e., $x \geq y \implies f(x) \geq f(y)$.

Correspondence $F : X \rightrightarrows X$ is **monotone** if $x \geq y \implies F(x) \geq_{SS} F(y)$.

Set of **fixed points** of self-correspondence F on X: $\mathcal{F}(F) := \{x \in X \mid x \in F(x)\}$.

Set of **fixed points** of self-map f on X: $\mathcal{F}(f) := \{x \in X \mid x = f(x)\}$.

Tarski's Fixed-Point Theorem

Theorem (Tarski 1955)

Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathcal{F}(f)$ is a non-empty complete lattice.

A whole new way to go about and find equilibria!

- Pause to appreciate: no constraints on what X is topologically (no metric space, no finite dimensional restriction, no convexity requirement).

  Purely order-theoretic.
- Why is this good? Because in many cases you could be working with weird spaces (policy functions, distributions, etc).
- More: you can choose whatever adequate $\geq$ if you only care about the existence.

And it's *not* just existence: $\mathcal{F}(f)$ is a non-empty complete lattice.

Tarski's fixed-point theorem gives structure to the set of fixed points.

Tarski's Fixed-Point Theorem

Theorem (Tarski 1955)

Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathcal{F}(f)$ is a non-empty complete lattice.

Full-blown proof is quite challenging.

- Guide to proof for Tarski's FPThm in Appendix to lecture notes. The full proof is quite sophisticated.
- The appendices to the lecture notes are for your reference only.

We will prove a more humble statement:

Lemma (Baby Tarski)

Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathcal{F}(f)$ is nonempty and has a largest element, $\bigvee_{\mathcal{F}(f)} \mathcal{F}(f)$.

Tarski's Fixed-Point Theorem

Lemma (Baby Tarski)

Let X be a complete lattice and f be a self-map on X. If f is monotone, then $\mathcal{F}(f)$ is nonempty and has a largest element, $\bigvee_{\mathcal{F}(f)} \mathcal{F}(f)$.

Proof

Let $S := \{x \in X : f(x) \geq x\}$. X is complete lattice $\implies f(\inf_X X) \geq \inf_X X \implies S \neq \emptyset$.

X is complete lattice and $S \subseteq X \implies y := \sup_X S \in X$.

Then, for any $x \in S$,

$$
\begin{aligned}
y \geq x &\implies f(y) \geq f(x) \geq x && \text{as } f \text{ is monotone and } x \in S \\
&\implies f(y) \geq y && \text{as } f(y) \text{ is an upper bound of } S \text{ and } y = \sup_X S \\
&\implies f(f(y)) \geq f(y) && \text{as } f \text{ is monotone} \\
&\implies f(y) \in S \\
&\implies y := \sup_X S \geq f(y) \\
&\implies y = f(y) && \text{by antisymmetry.} \quad \square
\end{aligned}
$$

Tarski-Zhou's Fixed-Point Theorem

Zhou (1994 GEB) generalised Tarski's FP theorem to monotone correspondences.

Low-hanging fruit with major significance and applications.

Theorem (Zhou 1994 GEB, Theorem 1)

Let X be a complete lattice and $F : X \rightrightarrows X$ be nonempty-valued. If F is monotone and, $\forall x \in X$, $F(x)$ is a complete sublattice, then $(\mathcal{F}(F), \geq)$ is nonempty complete lattice.

Hard to overstate the usefulness:

Think F as cartesian product of best-response mappings.

$\mathcal{F}(F)$ as set of Nash equilibria.

Tarski-Zhou's FPT says that Nash equilibria form a complete lattice!

Provides clear-cut way of talking about largest/smallest equilibria.

Weaker Monotone Mappings

Another useful fixed-point theorem based on weaker monotonicity properties.

Some preliminaries: For $S \in \mathbf{2}^X$, denote $F(S) := \cup_{x \in S} F(x)$.

A reminder:

Definition

(i) $S' \geq_{uws} S$ iff $\forall x \in S, \exists x' \in S'$ s.t. $x' \geq x$.

(ii) $S' \geq_{lws} S$ iff $\forall x' \in S', \exists x \in S$ s.t. $x' \geq x$.

(iii) $S' \geq_{ws} S$ iff $S' \geq_{uws} S$ and $S' \geq_{lws} S$.

Definition

(i) F is **upper weak set monotone** iff $F(x') \geq_{uws} F(x)\ \forall x' \geq x$.

(ii) F is **lower weak set monotone** iff $F(x') \geq_{lws} F(x)\ \forall x' \geq x$.

(iii) F is **weak set monotone** iff $F(x') \geq_{ws} F(x)\ \forall x' \geq x$.

(iv) F is **strong set monotone** iff $F(x') \geq_{ss} F(x)\ \forall x' \geq x$.

Weaker Monotone Mappings

Definition

(i) F is **upper weak set monotone** iff $F(x') \geq_{uws} F(x)\ \forall x' \geq x$.

(ii) F is **lower weak set monotone** iff $F(x') \geq_{lws} F(x)\ \forall x' \geq x$.

(iii) F is **weak set monotone** iff $F(x') \geq_{ws} F(x)\ \forall x' \geq x$.

(iv) F is **strong set monotone** iff $F(x') \geq_{ss} F(x)\ \forall x' \geq x$.

Roughly put:

Upper weak set monotonicity: with larger $x' > x$, for anything in $F(x)$, can find something larger in $F(x')$.

Note: If F always has a largest element, then F is uws monotone iff the largest element is increasing in x.

Lower weak set monotonicity: with smaller $x < x'$, for anything in $F(x')$, can find something smaller in $F(x)$.

Note: If F always has a smallest element, then F is lws monotone iff the smallest element is increasing in x.

Weaker Monotone Mappings

A property of weak set monotone correspondences for later:

Lemma (Che, Kim, & Kojima 2021 WP, Lemma 2)

Let $F : X \rightrightarrows Y$, where X, Y are posets. If F is weak set monotone, then for any subsets $S', S \subseteq X$ such that $S' \geq_{ws} S$, $F(S') \geq_{ws} F(S)$.

Li–Che–Kim–Kojima Fixed Point Theorem

Theorem (Li 2014; Che, Kim, & Kojima 2021 WP, Theorem 6)

Let X be a compact partially ordered metric space. Let $F : X \rightrightarrows X$ be a nonempty- and closed-valued correspondence on X.

(i) If F is upper ws monotone and $\exists x, y \in X : x \leq y \in F(x)$, then it has a maximal fixed point.

(ii) If F is lower ws monotone and $\exists x, y \in X : x \geq y \in F(x)$, then it has a minimal fixed point.

Non-nested conditions compared with Tarski-Zhou FPT. Weaker in many settings, but also deliver less.

Topological restrictions also crept back.

Also:

maximal $\equiv$ not dominated by anything else

$\not\equiv$ dominates everything else $\equiv$ supremum.

minimal $\equiv$ not dominating anything else

$\not\equiv$ dominated by everything else $\equiv$ infimum.

Proof beyond scope of this class.

Overview

1. Motivation
2. Ordering Sets – Again
3. Fixed-Point Theorems
4. Monotone Comparative Statics on Fixed Points
   - Monotone Comparative Statics on Fixed Points of Functions
   - Monotone Comparative Statics on Fixed Points of Correspondences
5. Games with Strategic Complementarities

Monotone Comparative Statics on Fixed Points

Eyes on the ball:

Before: Order-theoretic fixed points.

Now: MCS on fixed points.

Applications: MCS on equilibria.

Monotone Comparative Statics on Fixed Points of Functions

Villas-Boas (1997 JET) provides collection of very useful MCS results for functions.

First, result for decreasing functions.

Theorem (Villas-Boas 1997 JET, Theorem 3)

Let $(X, \geq)$ be a preordered set, and $f, g : X \to X$. If (i) $f \gg g$, (ii) $\forall x, y \in X : x \geq y \implies f(y) \geq f(x)$,
then $\forall x \in \mathcal{F}(f), y \in \mathcal{F}(g), \neg(y > x)$.

Proof

Suppose not, i.e., $\exists x \in \mathcal{F}(f), y \in \mathcal{F}(g) : y > x$. Then

(i) $f(y) > g(y) \because f \gg g$;

(ii) $g(y) = y \because y \in \mathcal{F}(g)$;

(iii) $y > x$, by assumption;

(iv) $x = f(x) \because x \in \mathcal{F}(f)$; and

(v) $f(x) \geq f(y) \because y > x \implies y \geq x \implies f(x) \geq f(y)$;

a contradiction.

Monotone Comparative Statics on Fixed Points of Functions

Theorem (Villas-Boas 1997 JET, Theorems 4 and 5)

Let X be poset, and $f, g : X \to X$.

(1) If (i) $\forall x \in X$, $X_{\geq x}$ is complete lattice, and (ii) f is weakly increasing, then $\forall$ fixed pt of g $\;x \in \mathcal{F}(g) : f(x) \geq\; (>)\, x$, $\exists$ fixed pt of f $\;y \in \mathcal{F}(f) : y \geq\; (>)\, x$.

If, in addition $f \geq\; (\gg)\, g$, then $\forall x \in \mathcal{F}(g)$, $\exists y \in \mathcal{F}(f) : y \geq\; (>)\, x$.

(2) If (i) $\forall x \in X$, $X_{x\leq}$ is complete lattice, and (ii) g is weakly increasing, then for every fixed pt of f $\;y \in \mathcal{F}(f) : y \geq\; (>)\, g(y)$, $\exists$ fixed pt of g $\;x \in \mathcal{F}(g) : y \geq\; (>)\, x$.

If in addition $f \geq\; (\gg)\, g$, then $\forall y \in \mathcal{F}(f)$, $\exists x \in \mathcal{F}(g) : y \geq\; (>)\, x$.

These are increadibly general and, therefore, increadibly useful.

We will prove (1); the proof for (2) is symmetric.

Monotone Comparative Statics on Fixed Points of Functions

Theorem (Villas-Boas 1997 JET, Theorems 4 and 5)

Let X be poset, and $f, g : X \to X$.

(1) If (i) $\forall x \in X, X_{\geq x}$ is complete lattice, and (ii) f is weakly increasing,
then $\forall$ fixed pt of g $\; x \in \mathcal{F}(g) : f(x) \geq \; (>) \, x$, $\exists$ fixed pt of f $\; y \in \mathcal{F}(f) : y \geq \; (>) \, x$.

If, in addition $f \geq \; (\gg) \, g$, then $\forall x \in \mathcal{F}(g), \exists y \in \mathcal{F}(f) : y \geq \; (>) \, x$.

Proof

- Let $x^* \in \mathcal{F}(g) : f(x^*) \geq \; (>) \, x^*$.
- By monotonicity, $f(x) \geq \; (>) \, x^* \; \forall x \in X_{\geq x^*}$.
- Let $\tilde{f} : X_{\geq x^*} \to X_{\geq x^*}$, where $\tilde{f}(x) = f(x)$.
  Now apply Tarki's fixed point theorem (for (i) what we've shown is enough) and conclude $\exists y \in X_{\geq x^*} : f(y) = \tilde{f}(y) = y \geq \; (>) \, x^*$.

Villas-Boas (1997 JET) also provides extensions for Banach spaces and correspondences under very general conditions.

useful for functional optimisation (e.g., solving for policy functions in macro, IO, etc.).

Monotone Comparative Statics on Fixed Points of Correspondences

Che, Kim, & Kojima (2021 WP) provide results for correspondences that are adjusted to the weak set order.

Theorem (Che, Kim, & Kojima 2021 WP, Theorem 7)

Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$.

(1) If (i) $\mathcal{F}(F) \neq \emptyset$, (ii) G is uws monotone, nonempty- and closed-valued, and (iii) $G(x) \geq_{uws} F(x)\ \forall x \in X$, then $\mathcal{F}(G) \geq_{uws} \mathcal{F}(F)$.

(2) If (i) $\mathcal{F}(G) \neq \emptyset$, (ii) F is lws monotone, nonempty- and closed-valued, and (iii) $G(x) \geq_{lws} F(x)\ \forall x \in X$, then $\mathcal{F}(G) \geq_{lws} \mathcal{F}(F)$.

Counterpart of Villas-Boas's Theorems 4 and 5.

Focus on proof for (1); proof for (2) is symmetric.

Monotone Comparative Statics on Fixed Points of Correspondences

Theorem (Che, Kim, & Kojima 2021 WP, Theorem 7)

Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$.

(1) If (i) $\mathcal{F}(F) \neq \emptyset$, (ii) G is uws monotone, nonempty- and closed-valued, and (iii) $G(x) \geq_{uws} F(x)$ $\forall x \in X$, then $\mathcal{F}(G) \geq_{uws} \mathcal{F}(F)$.

Proof

Fix any $x^* \in \mathcal{F}(F)$. $\forall S \subseteq X$, define (a) $S_{\geq x^*} := \{x \in S \mid x \geq x^*\}$;

(b) $S_+(F) := \{x \in S \mid \exists y \geq x \text{ s.t. } y \in F(x)\}$; (c) $S_-(F) := \{x \in S \mid \exists y \leq x \text{ s.t. } y \in F(x)\}$.

Let $\tilde{G}$ be self-correspondence on $X_{\geq x^*}$ s.t. $\tilde{G}(x) := G(x) \cap X_{\geq x^*}$ $\forall x \in X_{\geq x^*}$.

Next: Show uws dominance by proving $\tilde{G}$ verifies conditions to have fixed point in $X_{\geq x^*}$.

($X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.)

If so: $\exists y \in \tilde{G}(y) = G(y) \cap X_{\geq x^*} \implies y \in \mathcal{F}(G)$ and $y \geq x^* \in \mathcal{F}(F)$; done!

Monotone Comparative Statics on Fixed Points of Correspondences

Proof

WTS $X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.

(i) X poset $\implies X_{\geq x^*}$ poset. (immediate)

(ii) WTS $X_{\geq x^*}$ is compact.

$\forall S$ closed, $S_{\geq x^*}$ is also closed; X compact metric space, then $S_{\geq x^*}$ is compact.

$\implies X_{\geq x^*}$ is compact.

Monotone Comparative Statics on Fixed Points of Correspondences

Proof

WTS $X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.

(i) X poset $\implies X_{\geq x^*}$ poset.

(ii) $X_{\geq x^*}$ is compact.

(iii) WTS $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$.

$x^* \in F(x^*) \leq_{uws} G(x^*) \implies \exists y \in G(x^*) : y \geq x^*$, i.e., $y \in \tilde{G}(x^*) = G(x^*) \cap X_{\geq x^*}$.

Monotone Comparative Statics on Fixed Points of Correspondences

Proof

WTS $X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.

(i) X poset $\implies X_{\geq x^*}$ poset.

(ii) $X_{\geq x^*}$ is compact.

(iii) $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$.

(iv) WTS $\tilde{G}$ closed-valued.

$\because$ G is closed-valued and $X_{\geq x^*}$ is closed.

Monotone Comparative Statics on Fixed Points of Correspondences

Proof

WTS $X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.

(i) X poset $\implies X_{\geq x^*}$ poset.

(ii) $X_{\geq x^*}$ is compact.

(iii) $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$.

(iv) $\tilde{G}$ closed-valued.

(v) WTS $\tilde{G}$ nonempty-valued.

$\forall x \in X_{\geq x^*}, G(x) \geq_{uws} G(x^*) \geq_{uws} F(x^*) \ni x^*$.

$\implies \forall x \in X_{\geq x^*}, \exists y \in G(x) : y \geq x^*$.

$\implies y \in \tilde{G}(x) = G(x) \cap X_{\geq x^*}. \implies \tilde{G}(x) \neq \emptyset$.

Monotone Comparative Statics on Fixed Points of Correspondences

Proof

WTS $X_{\geq x^*}$ compact poset; $\tilde{G}$ nonempty- and closed-valued, and uws monotone.

(i) X poset $\implies X_{\geq x^*}$ poset.

(ii) $X_{\geq x^*}$ is compact.

(iii) $x^* \in X_+(\tilde{G}) \subseteq X_{\geq x^*}$.

(iv) $\tilde{G}$ closed-valued.

(v) $\tilde{G}$ nonempty-valued.

(vi) WTS $\tilde{G}$ is uws monotone.

G is uws monotone.

$\implies \forall x, x' \in X_{\geq x^*} : x' \geq x$, and $\forall y \in \tilde{G}(x) \subseteq G(x)$, $\exists y' \in G(x') : y' \geq y$.

$(y \geq x^* \text{ and } y' \geq y) \implies y' \geq x^*$.

$(y' \in G(x') \text{ and } y' \geq x^*) \implies y' \in G(x') \cap X_{\geq x^*} = \tilde{G}(x')$.

(vii) $\therefore$ satisfy conditions for $(\tilde{G}) \neq \emptyset$ as per Theorem 6 in Che, Kim, & Kojima 2021 WP.

Conclude $\mathcal{F}(G) \geq_{uws} \mathcal{F}(F)$.

Monotone Comparative Statics on Fixed Points of Correspondences

Theorem (Che, Kim, & Kojima 2021 WP, Theorem 7)

Let X be compact partially ordered metric space and $F, G : X \rightrightarrows X$.

(1) If (i) $\mathcal{F}(F) \neq \emptyset$, (ii) G is uws monotone, nonempty- and closed-valued, and (iii) $G(x) \geq_{uws} F(x)\ \forall x \in X$, then $\mathcal{F}(G) \geq_{uws} \mathcal{F}(F)$.

(2) If (i) $\mathcal{F}(G) \neq \emptyset$, (ii) F is lws monotone, nonempty- and closed-valued, and (iii) $G(x) \geq_{lws} F(x)\ \forall x \in X$, then $\mathcal{F}(G) \geq_{lws} \mathcal{F}(F)$.

In brief: upper/lower weak set dominance of correspondences (+ other conditions) implies upper/lower weak set dominance of their fixed points.

Corollary

Let X be a compact partially ordered metric space and $F, G : X \rightrightarrows X$. If (i) F and G are nonempty- and closed-valued, (ii) F is lws monotone, G is uws monotone, and (iii) $G(x) \geq_{ws} F(x)\ \forall x \in X$, then $\mathcal{F}(G) \geq_{ws} \mathcal{F}(F)$.

Overview

1. Motivation
2. Ordering Sets – Again
3. Fixed-Point Theorems
4. Monotone Comparative Statics on Fixed Points
5. Games with Strategic Complementarities
   - Games with Strong Strategic Complementarities
   - Supermodular Games
   - Games with Weak Strategic Complementarities

Games with Strategic Complementarities

Goal: provide general results similar to "if B_i increases, then set of equilibria increases."

Examples:

- If payoffs for player's action increase (e.g., subsidies), player chooses it more?
- It strategy space decreases (e.g., price caps, regulation, higher taxes), then affects deviations; will equilibrium 'decrease'?

Consider reduced-form games for more generally applicable results.

Games with Strategic Complementarities

Reduced-form game $G = \langle I, X, B \rangle$:

(i) Finite set players I.

(ii) Player i's strategy space X_i, $X = \times_{i \in I} X_i$.

(iii) Player i's behaviour, $B_i : X_{-i} \rightrightarrows X_i$; $B = (B_i)_{i \in I}$.

$B(x) : X \rightrightarrows X$, s.t. $B(x) := \times_{i \in I} B_i(x_{-i})$.

B summarises all components of reduced-form game (dfn of B depends on I and X).

Set of fixed points of G denoted by $\mathcal{F}(B)$: $\mathcal{F}(B) := \{x \in X \mid x_i \in B_i(x_{-i})\}$.

B_i may or may not be given as the best-response correspondence in a game,
i.e. $B_i(x_{-i}) = \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$.

results more general, can be applied to equilibrium models and solution concepts other than Nash equilibrium.

Games with Strong Strategic Complementarities

Use Tarski-Zhou's fixed point theorem to show an equilibrium exists:

Theorem

Let X_i be a complete lattice $\forall i \in I$.
If $B_i : X_{-i} \rightrightarrows X_i$ is strong set monotone and nonempty- and complete-sublattice-valued for every $i \in I$, then $\mathcal{F}(B)$ is non-empty and a complete lattice.

Games with Strong Strategic Complementarities

MCS on equilibria:

Theorem

Let $X_i, \tilde{X}_i$ be complete lattices wrt same partial order, and $\tilde{X}_i \geq_{ss} X_i, \forall i \in I$.
Let $B_i : X_{-i} \rightrightarrows X_i$ and $\tilde{B}_i : \tilde{X}_{-i} \rightrightarrows \tilde{X}_i$ be strong set monotone and nonempty- and complete-sublattice-valued $\forall i \in I$.
If $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})\ \forall i \in I$ and $\forall x_{-i} \in X_{-i}, \tilde{x}_{-i} \in \tilde{X}_{-i} : \tilde{x}_{-i} \geq x_{-i}$,
then (1) $\mathcal{F}(\tilde{B}), \mathcal{F}(B)$ are nonempty complete lattices, and (2) $\sup_{\mathcal{F}(\tilde{B})} \mathcal{F}(\tilde{B}) \geq \sup_{\mathcal{F}(B)} \mathcal{F}(B)$ and $\inf_{\mathcal{F}(\tilde{B})} \mathcal{F}(\tilde{B}) \geq \inf_{\mathcal{F}(B)} \mathcal{F}(B)$.

Result implies $\mathcal{F}(\tilde{B}) \geq_{ws} \mathcal{F}(B)$.

Games with Strong Strategic Complementarities

Proof

Know that $\forall i$: $X_i, \tilde{X}_i$ be complete lattices; $\tilde{X}_i \geq_{ss} X_i$; $B_i, \tilde{B}_i$ ss monotone, nonempty- and complete-sublattice-valued; $\tilde{x}_{-i} \geq x_{-i} \implies \tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$.

- (1) $\mathcal{F}(\tilde{B}), \mathcal{F}(B)$ are nonempty complete lattices (by Zhou-Tarski FPT); admit largest and smallest element.
- (2) WTS largest fixed pt of $\tilde{B}$ greater than largest fixed pt of B; proof symmetric for smallest.

Let $b_i^*(x_{-i}) := \sup_{X_i} B_i(x_{-i})$ and $b_{i*}(x_{-i}) := \inf_{X_i} B_i(x_{-i})$.

B_i complete-sublattice-valued $\implies b_i^*(x_{-i}), b_{i*}(x_{-i}) \in B_i(x_{-i})$.

$b^*(x) := \sup_X B(x) \equiv (b_i^*(x_{-i}))_{i \in I}$; and $b_*(x) := \inf_X B(x) \equiv (b_{i*}(x_{-i}))_{i \in I}$. Define $\tilde{b}^*$ and $\tilde{b}_*$ analogously, on $\tilde{X}$.

Claim: Largest (smallest) fixed pt of b^* (b_*) is the largest (smallest) fixed pt of B. (Proof left as an exercise.)

Claim: $\tilde{b}^*(\tilde{x}) \geq b^*(x)$ for any $\tilde{x} \geq x$.

$\tilde{x} \geq x \implies \tilde{x}_{-i} \geq x_{-i} \forall i \implies \tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i}) \forall i \implies \tilde{B}(\tilde{x}) \geq_{ss} B(x)$.

As $B_i, \tilde{B}_i$ ss monotone and nonempty- and complete-sublattice-valued,

Games with Strong Strategic Complementarities

Proof

Know that $\forall i$: $X_i, \tilde{X}_i$ be complete lattices; $\tilde{X}_i \geq_{ss} X_i$; $B_i, \tilde{B}_i$ ss monotone, nonempty- and complete-sublattice-valued; $\tilde{x}_{-i} \geq x_{-i} \implies \tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$.

- WTS largest fixed pt of $\tilde{B}$ greater than largest fixed pt of B; proof symmetric for smallest.

$b^*(x) \equiv (b_i^*(x_{-i}))_{i \in I}$; and $b_*(x) \equiv (b_{i*}(x_{-i}))_{i \in I}$. Define $\tilde{b}^*$ and $\tilde{b}_*$ analogously, on $\tilde{X}$.

Claim: Largest (smallest) fixed pt of b^* (b_*) is the largest (smallest) fixed pt of B.

Claim: $\tilde{b}^*(\tilde{x}) \geq b^*(x)$ for any $\tilde{x} \geq x$.

Claim: $\tilde{b}^*$ is monotone.

If $X = \tilde{X}$, as $X_{\geq x}$ is complete lattice $\forall x$, use Villas-Boas (1997 JET), Theorems 4 and 5.

In general, can have $X \neq \tilde{X}$; we will need an extra step.

Proof

Know that $\forall i$: $X_i, \tilde{X}_i$ be complete lattices; $\tilde{X}_i \geq_{SS} X_i$; $B_i, \tilde{B}_i$ ss monotone, nonempty- and complete-sublattice-valued; $\tilde{x}_{-i} \geq x_{-i} \implies \tilde{B}_i(\tilde{x}_{-i}) \geq_{SS} B_i(x_{-i})$.

- WTS largest fixed pt of $\tilde{B}$ greater than largest fixed pt of B; proof symmetric for smallest.

$b^*(x) \equiv (b_i^*(x_{-i}))_{i \in I}$; and $b_*(x) \equiv (b_{i*}(x_{-i}))_{i \in I}$. Define $\tilde{b}^*$ and $\tilde{b}_*$ analogously, on $\tilde{X}$.

Claim: Largest (smallest) fixed pt of b^* (b_*) is the largest (smallest) fixed pt of B.

Claim: $\tilde{b}^*(\tilde{x}) \geq b^*(x)$ for any $\tilde{x} \geq x$ **Claim**: $\tilde{b}^*$ is monotone.

Claim: Let x^* and $\tilde{x}^*$ be the largest fixed points of b^* and $\tilde{b}^*$. Then $\tilde{x}^* \geq x^*$.

Let $\tilde{X}_{\geq x^*} := \{x \in \tilde{X} \mid x \geq x^*\}$.

$\tilde{X} \geq_{SS} X \implies \forall x \in \tilde{X}, x^* \in X, x \vee x^* \in \tilde{X}$; hence $\tilde{X}_{\geq x^*} \neq \emptyset$.

$\tilde{X}$ complete lattice $\implies$ so is $\tilde{X}_{\geq x^*}$. Define $\tilde{g}^*$ on $\tilde{X}_{\geq x^*}$ as $\tilde{g}^*(x) = \tilde{b}^*(x)$.

As $\forall x \in \tilde{X}_{\geq x^*}, x \geq x^*$, then $\tilde{g}^*(x) = \tilde{b}^*(x) \geq b^*(x^*)$.

Hence $\tilde{g}^*(x) \in \tilde{X}_{\geq x^*}$ $\forall x \in \tilde{X}_{\geq x^*}$, and $\tilde{g}^*$ is a self-map on a complete lattice.

As $\tilde{b}^*$ is monotone, so is $\tilde{g}^*$.

By Tarski's fixed point theorem, $\exists y^* \in \tilde{X}_{\geq x^*} : y^* = \tilde{g}^*(y^*) = \tilde{b}^*(y^*) \geq x^* = b^*(x^*)$.

Games with Strong Strategic Complementarities

Let's go back to normal-form games $\Gamma = \langle I, X, u \rangle$.

Define B_i as player i's best-response correspondence: $B_i(x_{-i}) := \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$.

Given Γ and $\tilde{\Gamma}$, what do we need to guarantee that

(i) $\tilde{B}_i, B_i$ are ss monotone,

(ii) $\tilde{B}_i(\tilde{x}_{-i}) \geq_{ss} B_i(x_{-i})$ for every $\tilde{x} \geq x$, and

(iii) $\tilde{B}_i, B_i$ are nonempty- and complete-sublattice-valued?

We already know the answer...

Games with Strong Strategic Complementarities

Theorem (Milgrom & Shannon 1994, Theorem 4)

Let X be a lattice and v, u be two real-valued functions on X. v and u are quasisupermodular and v single-crossing dominates u if and only if, for $S' \geq_{ss} S$, $X(S'; v) \geq_{ss} X(S; u)$.

Corollary (Milgrom & Shannon 1994, Corollary 2)

Let X be a lattice, S a sublattice, and $f : X \to \mathbb{R}$. If f is quasisupermodular, then $X(S; f)$ is a sublattice of S.

$X_i, \tilde{X}_i$ be (i) compact, and complete sublattices of a lattice Y_i, and (ii) $\tilde{X}_i \geq_{ss} X_i$.

$u_i, \tilde{u}_i$ be (i) quasisupermodular in $(x_i; x_{-i})$ and $(\tilde{x}_i; \tilde{x}_{-i})$ (resp.), (ii) continuous in x_i and $\tilde{x}_i$ (resp.); and (iii) $\tilde{u}_i \geq_{SC} u_i$.

Overview

Supermodular Games

Existing literature focuses on changes u.

Definition

A class of games $\{\Gamma(t)\}_{t \in T}$ has **strategic complementarities** if $\Gamma(t) = \langle I, X, u^t \rangle$, where I is finite, T is a poset, and, for all i, X_i is a compact lattice, $u_i^t : X \to \mathbb{R}$ is continuous and quasisupermodular in x_i and satisfies the single-crossing property in $(x_i; x_{-i}, t)$.

Can weaken continuity of u_i^t with upper semi-continuity in x_i and continuity in x_{-i}, separately.

These are also called **supermodular games**.

Supermodular Games

Theorem (Milgrom & Roberts 1990 Ecta; Milgrom & Shannon 1994 Ecta)

Let $\{\Gamma(t)\}_{t \in T}$ have strategic complementarities. For any t, let $X^{NE}(t)$ denote the set of pure Nash equilibria of $\Gamma(t)$.
$X^{NE}(t)$ is a complete lattice, monotone wrt t in the strong set order.
Furthermore, for any t, the largest and and smallest Nash equilibria are the largest and smallest outcomes (resp.) survives IESDS.

Corollary

(a) A supermodular game has a pure strategy Nash equilibrium

(b) The greatest and least strategy profiles in the sets of (i) strategy profiles surviving IESDS, (ii) rationalisable strategy profiles, (iii) correlated equilibria, and (iv) Nash equilibria exist and are all the same.

(c) If a supermodular game has a unique Nash equilibrium, it is dominance solvable.

Not only of theoretical but also of practical interest.

Obtain the greatest and smallest PSNE via simple iterative operator.

Games with Weak Strategic Complementarities

We conclude, briefly, by provide weak set order counterparts to our previous results.

Definition

A reduced-form normal-form game G has **upper (resp. lower) weak strategic complementarities** if

(i) $\exists x \in X : y_i \in B_i(x_{-i})$ and $\exists y_i \in X_i$ s.t. $y_i \geq x_i$ (resp. $\leq$) $\forall i$;

(ii) B_i is uws (resp. lws) monotone;

(iii) $B_i : X_{-i} \rightrightarrows X_i$ nonempty- and compact-valued, $\forall i$; and

(iv) X_i is a compact partially ordered metric space.

Theorem (Che, Kim, & Kojima 2021 WP, Theorem 9(i))

Let G be a reduced-form normal-form game.
If G has upper or lower weak strategic complementarities, then the set of fixed points of B, $\mathcal{F}(B)$, is nonempty.

Proof

Follows from Li–Che–Kim–Kojima Fixed Point Theorem.

Games with Weak Strategic Complementarities

Theorem (Che, Kim, & Kojima 2021 WP, Theorem 9(ii))

Let $G, \tilde{G}$ be two reduced-form normal-form games.

- If $\mathcal{F}(B) \neq \emptyset$, $\tilde{G}$ has upper weak strategic complementarities, and $\tilde{B}_i(s_{-i}) \geq_{uws} B_i(s_{-i})$ $\forall s_{-i}, \forall i$, then $\mathcal{F}(\tilde{B}) \geq_{uws} \mathcal{F}(B)$.
- If $\mathcal{F}(\tilde{B}) \neq \emptyset$, G has lower weak strategic complementarities, and $\tilde{B}_i(s_{-i}) \geq_{lws} B_i(s_{-i})$ $\forall s_{-i}, \forall i$, then $\mathcal{F}(\tilde{B}) \geq_{lws} \mathcal{F}(B)$.

Proof

Follows from weak MCS results.